PROJECTIVE MANIFOLDS CONTAINING A LARGE LINEAR SUBSPACE WITH NEF NORMAL BUNDLE

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Abstract. We classify smooth complex projective varieties $X \subset \mathbb{P}^N$ of dimension $2s+1$ containing a linear subspace Λ of dimension s whose normal bundle $N_{\Lambda/X}$ is numerically effective.

1. Introduction

Let $X \subset \mathbb{P}^N$ be a smooth complex projective variety of dimension n, containing a linear subspace Λ of dimension s; denote by $N_{\Lambda/X}$ its normal bundle and by c the degree of the first Chern class of $N_{\Lambda/X}$.
If $N_{\Lambda/X}$ is numerically effective, then X is covered by lines; if furthermore s is sufficiently large, the restrictions imposed on X become stronger.

By [9, Theorem 2.5], if $s+c > \frac{n}{2}$ then, for large m, denoting by H the restriction to X of the hyperplane bundle, the linear system $|m(K_X + (s+1+c)H)|$ defines an extremal ray contraction of X which contracts Λ. If s is greater than $\frac{n}{2}$ this contraction is a projective bundle, as shown in [29] (see also [7, Theorem 2.5]). The same result holds if $s = \frac{n}{2}$ and $N_{\Lambda/X}$ is trivial, by [13, Theorem 1.7] and [33, Theorem 2.4].

The complete study of the case $s = \frac{n}{2}$ is the subject of [29]; the setup of the quoted paper is different - it is not assumed the existence of a linear space of dimension $\frac{n}{2}$ with nef normal bundle, but the existence of a linear space of dimension $\frac{n}{2}$ through every point of X - yet the assumptions are in fact equivalent.
The most difficult cases in [29] are manifolds of Picard number one, which turn out to be, besides linear spaces, hyperquadrics and Grassmannians of lines.

In this paper we study the next case, i.e. $n = 2s+1$, proving the following

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$, containing a linear subspace Λ of dimension s, such that its normal bundle $N_{\Lambda/X}$ is numerically effective. If the Picard number of X is one, then X is one of the following:*

1. *a linear space $\mathbb{P}^{2s+1}$;*
2. *a smooth hyperquadric $\mathbb{Q}^{2s+1}$;*
3. *a cubic threefold in $\mathbb{P}^4$;*
4. *a complete intersection of two hyperquadrics of dimension 4 in $\mathbb{P}^5$;*
5. *the intersection of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with three general hyperplanes;*
6. *a hyperplane section of the Grassmannian of lines $\mathbb{G}(1,s+2)$ in its Plücker embedding.*

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If the Picard number of X is greater than one, then there is an elementary contraction $\varphi\colon X \to Y$ which contracts Λ and one of the following occurs:

(7) *$\varphi\colon X \to Y$ is a scroll;*
(8) *Y is a curve, and the general fiber of φ is*
 (8a) *the Grassmannian of lines $\mathbb{G}(1, s)$;*
 (8b) *a smooth hyperquadric $\mathbb{Q}^{2s}$;*
 (8c) *a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.*

The outline of the paper is the following: first of all, we use the theory of uniform vector bundles on the projective space, together with some standard exact sequence, to classify all possible normal bundles $N_{\Lambda/X}$. Then we consider separately the case of Picard number one and the case of greater Picard number; in fact the ideas and the proofs are very different.

If the Picard number is one, the main idea is to study the manifold $\widetilde{X}$ obtained by blowing-up X along Λ; we prove that $\widetilde{X}$ is a Fano manifold, and then we study its "other" extremal contraction. As a first application of this construction, in section (5) we show how to use it to complete [29, Main Theorem].
In the setup of Theorem (1.1), the hardest case corresponds to the normal bundle $N_{\Lambda/X} \simeq T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$, which give rise to case (6). In this case we need to use twice the blow-up construction: first we blow-up X along Λ and we show that there is a special one-parameter family Σ of linear spaces to which Λ belongs; then we blow-up X along Σ and, studying this blow-up, we are able to describe completely the variety.

If the Picard number is greater than one, we combine the ideas and techniques of [9] with those of [10] to show that a dominating family of lines on X of anticanonical degree $\geq \frac{n+1}{2}$ is extremal, i.e., the numerical class of a line spans a Mori extremal ray of $\mathrm{NE}(X)$. The contraction of this ray is the morphism $\varphi\colon X \to Y$ appearing in the second part of the statement of Theorem (1.1). The general fiber F of φ is then a manifold covered by linear spaces of dimension $\geq \frac{\dim F}{2}$, and this leads to its classification.

2. Background material

A smooth complex projective variety X is called *Fano* if its anticanonical bundle $-K_X$ is ample; the *index* r_X of X is the largest natural number such that $-K_X = mH$ for some (ample) divisor H on X. Since X is smooth, $\mathrm{Pic}(X)$ is torsion free, therefore the divisor L satisfying $-K_X = r_X L$ is uniquely determined and called the *fundamental divisor* of X. Fano manifolds with $r_X = \dim X - 1$ are called *del Pezzo* manifolds.

2.1. **Extremal contractions.** Let X be a smooth projective variety of dimension n defined over the field of complex numbers; a *contraction* $\varphi\colon X \to Z$ is a proper surjective map with connected fibers onto a normal variety Z.
If the canonical bundle K_X is not nef, then the negative part of the cone $\mathrm{NE}(X)$ of effective 1-cycles is locally polyhedral, by the Cone Theorem. By the Contraction Theorem, to every face in this part of the cone is associated a contraction, called *extremal contraction* or *Fano–Mori contraction*.
An extremal contraction associated to a face of dimension one, i.e. to an extremal

ray, is called an *elementary contraction*; a Cartier divisor H such that $H = \varphi^*A$ for an ample divisor A on Z is called a *supporting divisor* of the contraction φ.

Definition 2.1.1. An elementary fiber type extremal contraction $\varphi\colon X \to Z$ is called a *scroll* (respectively a *quadric fibration*) if there exists a φ-ample line bundle $L \in \mathrm{Pic}(X)$ such that $K_X + (\dim X - \dim Z + 1)L$ (respectively $K_X + (\dim X - \dim Z)L$) is a supporting divisor of φ.
An elementary fiber type extremal contraction $\varphi\colon X \to Z$ onto a smooth variety Z is called a $\mathbb{P}$-*bundle* (respectively *quadric bundle*) if there exists a vector bundle $\mathcal{E}$ of rank $\dim X - \dim Z + 1$ (respectively of rank $\dim X - \dim Z + 2$) on Z such that $X \simeq \mathbb{P}_Z(\mathcal{E})$ (respectively there exists an embedding of X over Z as a divisor of $\mathbb{P}_Z(\mathcal{E})$ of relative degree 2).
Some special scroll contractions arise from projectivization of Bănică sheaves (cf. [6]); in particular, if $\varphi\colon X \to Z$ is a scroll such that every fiber has dimension $\leq \dim X - \dim Z + 1$, then Z is smooth and X is the projectivization of a Bănică sheaf on Z (cf. [6, Proposition 2.5]); we will call these contractions *special Bănică scrolls*.

2.2. **Families of rational curves.**

Definition 2.2.1. A *family of rational curves* is an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ (see [22, Definition 2.11]). We will say that V is *unsplit* if it is proper.
We define $\mathrm{Locus}(V)$ to be the set of points of X through which there is a curve among those parametrized by V and we say that V is a *dominating family* if $\overline{\mathrm{Locus}(V)} = X$. We denote by V_x the subscheme of V parametrizing rational curves among those parametrized by V passing through x. By abuse of notation, given a line bundle $L \in \mathrm{Pic}(X)$, we will denote by $L \cdot V$ the intersection number $L \cdot C_V$, with C_V any curve among those parametrized by V.

Definition 2.2.2. An unsplit dominating family V defines a relation of rational connectedness with respect to V, which we shall call *rc*(V)*-relation* for short, in the following way: x and y are in rc(V)-relation if there exists a chain of rational curves among those parametrized by V which joins x and y.

To the rc(V)-relation we can associate a fibration, at least on an open subset ([11], [22, IV.4.16]); we will call it *rc*(V)*-fibration*.

Proposition 2.2.3. [22, IV.2.6] *Let X be a smooth projective variety, let V be an unsplit family of rational curves on X and let x be a point in* $\mathrm{Locus}(V)$*. Then*

(a) $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
(b) $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

This last proposition, in case V is the unsplit family of deformations of a minimal extremal rational curve, gives the *fiber locus inequality*:

Proposition 2.2.4. [19, Theorem 0.4], [32, Theorem 1.1] *Let X be a smooth complex projective variety and let φ be a Fano–Mori contraction of X. Let $E := \mathrm{Exc}(\varphi)$ be the exceptional locus of φ and let F be an irreducible component of a (non-trivial) fiber of φ. Then*

$$\dim E + \dim F \geq \dim X + l - 1,$$

where $l := \min\{-K_X \cdot C \mid C$ is a rational curve in $F\}$. If φ is the contraction of an extremal ray R, then $l(R) := l$ is called the length of the ray.

Definition 2.2.5. Let X be a smooth variety, V an unsplit families of rational curves on X and $Z \subset X$. We denote by $\mathrm{Locus}(V)_Z$ the set of points $x \in X$ such that there exists a curve C in V with $C \cap Z \neq \emptyset$ and $x \in C$.

We will use some properties of $\mathrm{Locus}(V)_Z$, summarized in the following

Lemma 2.2.6. [12, Section 2], [9, Proof of Lemma 1.4.5] *Let $Z \subset X$ be a closed subset and V an unsplit family. Assume that curves contained in Z are numerically independent from curves in V, and that $Z \cap \mathrm{Locus}(V) \neq \emptyset$. Then*

$$\dim \mathrm{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1.$$

If σ is an extremal face of $\mathrm{NE}(X)$, F is a fiber of the contraction associated to σ and V is an unsplit family, numerically independent from curves whose numerical class is in σ, then

$$\mathrm{NE}^X(\mathrm{Locus}(V)_F) = \langle \sigma, [V] \rangle,$$

i.e., the numerical class in X of a curve in $\mathrm{Locus}(V)_F$ is in the subcone of $\mathrm{NE}(X)$ generated by σ and $[V]$.

2.3. **Some extremal contractions related to Grassmannians.** We will now present some examples of Fano manifolds admitting a projective bundle structure and another extremal contraction φ whose target is a Grassmannian of lines. We will use these descriptions later in our proofs.

Example 2.3.1. Let $\mathbb{G}(1,s)$ be the Grassmannian of lines in $\mathbb{P}^s$ and denote by $\mathcal{I}$ the incidence variety. Consider the incidence diagram:

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "p"'] \arrow[dr, "\varphi"] & \\ \mathbb{P}^s & & \mathbb{G}(1,s). \end{tikzcd}$$

Then p and φ are projective bundles, namely $\mathcal{I} = \mathbb{P}_{\mathbb{P}^s}(p_*\varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(1)) = \mathbb{P}_{\mathbb{P}^s}(\Omega_{\mathbb{P}^s}(2))$ and $\mathcal{I} = \mathbb{P}_{\mathbb{G}(1,s)}(\varphi_* p^*\mathcal{O}_{\mathbb{P}^s}(1)) = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q})$, where $\mathcal{Q}$ is the universal quotient bundle on $\mathbb{G}(1,s)$.

Example 2.3.2. As in the previous example, let $\mathbb{G}(1,s)$ be the Grassmannian of lines in $\mathbb{P}^s$ and let $\mathcal{I}$ denote the incidence variety; consider the following diagram, obtained by the incidence diagram:

$$\begin{tikzcd} & \mathcal{I} \times \mathbb{P}^1 \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[drr, bend left, "\varphi"] & & \\ \mathbb{P}^s \times \mathbb{P}^1 & & \mathbb{G}(1,s) \times \mathbb{P}^1 \arrow[r, "g"] & \mathbb{G}(1,s). \end{tikzcd}$$

The composition $\varphi = g \circ q$ gives a morphism $\varphi \colon \mathcal{I} \times \mathbb{P}^1 \to \mathbb{G}(1,s)$ whose fibers are smooth two-dimensional quadrics. Let $\mathcal{H}$ be $p^*\mathcal{O}_{\mathbb{P}^s \times \mathbb{P}^1}(1,1)$ and put $\mathcal{E} := \varphi_*\mathcal{H}$. We have

$$\mathcal{E} = \varphi_*\mathcal{H} = g_*(q_*\mathcal{H}) = g_*(\mathcal{O}_{\mathbb{G}(1,s)\times\mathbb{P}^1}(1,0) \otimes g^*\mathcal{Q}) = \mathcal{Q}^{\oplus 2}.$$

The product $\mathcal{I} \times \mathbb{P}^1 = \mathbb{P}_{\mathbb{P}^s \times \mathbb{P}^1}(p_1^*\Omega_{\mathbb{P}^s}(2))$, where p_1 denotes the projection onto $\mathbb{P}^s$, embeds in $\mathbb{P} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ as a divisor of relative degree 2, i.e. it belongs to a linear system $|2\mathcal{H} - \varphi^*L|$ for some line bundle L in $\mathrm{Pic}(\mathbb{G}(1,s))$. The discriminant divisor of the quadric bundle is in the linear system $|2\det\mathcal{E} - 4L|$ and it is trivial, since every fiber of φ is smooth. It follows that $L = \mathcal{O}_{\mathbb{G}(1,s)}(1)$.

Example 2.3.3. Let $\mathbb{G}(1, s+1)$ be the Grassmannian of lines in $\mathbb{P}^{s+1}$ and let $\mathbb{G}(1, H) \subset \mathbb{G}(1, s+1)$ be a sub-Grassmannian corresponding to the lines of $\mathbb{P}^{s+1}$ contained in a fixed hyperplane H.
Consider the rational map $\psi\colon \mathbb{G}(1, s+1) \dashrightarrow H$, which associates to a line l the point of intersection of l with H. This map is not defined precisely along the points of $\mathbb{G}(1, s+1)$ representing the lines contained in H, i.e., along the sub-Grassmannian $\mathbb{G}(1, H)$.
Consider the resolution of φ, obtained by blowing-up $\mathbb{G}(1, s+1)$ along $\mathbb{G}(1, H)$:

$$\begin{tikzcd} \widetilde{\mathbb{G}}(1, s+1) \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{G}(1, s+1) \arrow[dl, dashed, "\psi"] \\ H & \end{tikzcd}$$

The contraction p is a $\mathbb{P}^{s+1}$-bundle over H, whose fibers are the strict transforms of linear subspaces $\mathbb{P}^{s+1} \subset \mathbb{G}(1, s+1)$ corresponding to stars of lines with center in H, namely $\widetilde{\mathbb{G}}(1, s+1) = \mathbb{P}_H(p_*\varphi^*\mathcal{O}_{\mathbb{G}(1,s+1)}(1)) = \mathbb{P}_H(\Omega_{\mathbb{P}^s}(2) \oplus \mathcal{O}_{\mathbb{P}^s}(1))$.

3. A general construction

In this section we will present a blow-up construction and we will show how to apply it to manifolds of Picard number one containing a linear space with nef normal bundle.
The construction in the following proposition has been inspired by the graduate thesis [27] supervised the second named author.

Proposition 3.1. *Let $X \subset \mathbb{P}^N$ be a Fano manifold of dimension n, index r_X and Picard number one, covered by lines and containing a smooth subvariety Σ of dimension s which is the intersection of its linear span with X, i.e., $\Sigma = X \cap \langle \Sigma \rangle$, with $\left[\frac{n}{2}\right] \leq s \leq n-2$. Let $\sigma\colon \widetilde{X} \to X$ be the blow-up of X along Σ and let $E = \mathbb{P}_\Sigma(N^*_{\Sigma/X})$ be the exceptional divisor of σ.*
Then $\mathrm{NE}(\widetilde{X}) = \langle [C_\sigma], [\ell] \rangle$, where C_σ is a minimal curve contracted by σ and ℓ is the strict transform of a line meeting Σ at one point.
If $r_X \geq \left[\frac{n}{2}\right] + 1$, then $\widetilde{X}$ is a Fano manifold. Moreover, the extremal contraction $\varphi\colon \widetilde{X} \to Y$ different from σ is defined by the linear system $|\sigma^*\mathcal{O}_X(1) - E|$ and the extremal ray associated with φ has length $r_X - n + s + 1$.
If $r_X \geq \left[\frac{n+1}{2}\right] + 1$, then also E is a Fano manifold. Moreover, the restriction to E of φ is the morphism given by the linear system $|\xi_{N^*_{\Sigma/X}(1)}|$.

Proof. Since the Picard number of X is one and X is covered by lines, it follows by [4, Proposition 1.1] that X is rationally connected with respect to a dominating family of lines.
Consider the rational map $X \dashrightarrow Y$ defined by the linear system $|\mathcal{O}_X(1) \otimes \mathcal{I}_\Sigma|$ of hyperplanes containing Σ. Let $\varphi\colon \widetilde{X} \to Y$ be the resolution of this map. Then the morphism φ is defined by the linear system $|\mathcal{H} - E|$, where $\mathcal{H}$ denotes by the pull-back $\sigma^*\mathcal{O}_X(1)$.
Let $l \subset X$ be a line meeting Σ but not contained in it; notice that such a line exists because X is rationally connected with respect to a family of lines.
Since $\Sigma = X \cap \langle \Sigma \rangle$, the intersection $l \cap \Sigma$ consists of one point, hence the morphism

φ contracts ℓ, the strict transform of l. Therefore, denoting by C_σ a rational curve of minimal degree contracted by σ, we obtain $\mathrm{NE}(\widetilde{X}) = \langle [C_\sigma], [\ell] \rangle$.
By the canonical bundle formula for blow-ups we have

$$-K_{\widetilde{X}} = -\sigma^* K_X - (n-s-1)E = r_X\mathcal{H} - (n-s-1)E. \tag{3.1.1}$$

Clearly, $-K_{\widetilde{X}} \cdot C_\sigma > 0$.

If $r_X \geq \left[\frac{n}{2}\right] + 1$, we get $-K_{\widetilde{X}} \cdot \ell = r_X - n + s + 1 > 0$. By the Kleiman criterion it follows that $-K_{\widetilde{X}}$ is ample, so $\widetilde{X}$ is a Fano manifold. We also get that the length of the ray contracted by φ is $r_X - n + s + 1$.

Assume now that $r_X \geq \left[\frac{n+1}{2}\right] + 1$. From (3.1.1) it follows that the line bundle

$$-K_{\widetilde{X}} - E = r_X\mathcal{H} - (n-s)E$$

is ample on $\widetilde{X}$, since $(-K_{\widetilde{X}} - E) \cdot C_\sigma = n - s$ and $(-K_{\widetilde{X}} - E) \cdot \ell = r_X + s - n > 0$. Therefore its restriction to E, which by adjunction is $-K_E$, is ample, too; hence E is a Fano manifold.
Denote by D the divisor $-K_{\widetilde{X}} - E + (\mathcal{H} - E)$. Then D is ample on $\widetilde{X}$, being the sum of an ample line bundle and a nef one, so $h^1(\mathcal{H} - 2E) = h^1(K_{\widetilde{X}} + D) = 0$, by the Kodaira Vanishing theorem. It follows that the morphism

$$H^0(\widetilde{X}, \mathcal{H} - E) \longrightarrow H^0(E, (\mathcal{H} - E)|_E) = H^0(E, \xi_{N^*_{\Sigma/X}(1)})$$

is surjective and the last claim is proved. □

The following lemma shows that we can apply our construction to manifolds of Picard number one containing a large linear space whose normal bundle is numerically effective.

Lemma 3.2. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and Picard number one, containing a linear space Λ of dimension s with $\left[\frac{n}{2}\right] \leq s \leq n-2$. Assume that the normal bundle $N_{\Lambda/X}$ of Λ in X is nef and denote by c the non negative integer such that $\det N_{\Lambda/X} = \mathcal{O}_\Lambda(c)$.*
Then X is a Fano manifold of index $r_X = s + 1 + c$ covered by lines.

Proof. By the adjunction formula we have

$$K_\Lambda = (K_X + \det N_{\Lambda/X})|_\Lambda,$$

whence

$$(-K_X)|_\Lambda = \mathcal{O}_\Lambda(s + 1 + c),$$

from which we can derive

$$-K_X = \mathcal{O}_X(s + 1 + c). \tag{3.2.1}$$

Let l be a general line in Λ. From the nefness of $N_{\Lambda/X}$ and the exact sequence

$$0 \longrightarrow N_{l/\Lambda} = \mathcal{O}_\Lambda(1)^{\oplus(s-1)} \longrightarrow N_{l/X} \longrightarrow (N_{\Lambda/X})|_l \longrightarrow 0,$$

we have that $N_{l/X}$ is nef; therefore l is a free rational curve in X (see [22, Definition II.3.1]), which is thus covered by lines by [22, Proposition II.3.10]. □

4. Normal bundles

Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$, containing a linear subspace Λ of dimension s whose normal bundle $N_{\Lambda/X}$ is numerically effective.
In this section we will start the proof of Theorem (1.1), giving the list of all possible normal bundles of the linear subspace Λ and showing that X is covered by linear subspaces of dimension s.

Proposition 4.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$, containing a linear subspace Λ of dimension s whose normal bundle $N_{\Lambda/X}$ is nef. Then $N_{\Lambda/X}$ is one of the following:*

(1) $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda$;
(2) $\Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)$;
(3) $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda(1)$;
(4) $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$;
(5) $\mathcal{O}(1)_\Lambda^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus(s+1-c)}$.

Moreover, through every point of X there is a linear subspace of dimension s.

Proof. From the exact sequence

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathbb{P}^N} = \mathcal{O}_\Lambda(1)^{\oplus(N-s)} \longrightarrow (N_{X/\mathbb{P}^N})|_X \longrightarrow 0$$

and the nefness of $N_{\Lambda/X}$ we get that the splitting of $N_{\Lambda/X}$ on lines in Λ is of type $(0, \dots, 0, 1, \dots, 1)$, hence uniform.
By the classification of uniform vector bundles of rank $s+1$ on $\mathbb{P}^s$ given in [15] and [5], taking into account the splitting type, we have that $N_{\Lambda/X}$ is one of the bundles listed in the statement. Notice that all these bundles are generated by global sections and have $h^1(N_{\Lambda/X}) = 0$.
It follows that the Hilbert scheme of s-planes in X is smooth at the point λ corresponding to Λ. Let T be the unique irreducible component of the Hilbert scheme containing λ and let Z be the universal family; we have the following diagram:

$$\begin{tikzcd} & Z \arrow[dl, "q"'] \arrow[dr, "p"] & \\ T & & X. \end{tikzcd}$$

Let z be a point in $\Lambda' := q^{-1}(\lambda)$; we consider the differential of p at that point

$$d_z p \colon T_z Z \longrightarrow T_{p(z)} X;$$

this map is the identity when restricted to $T_z \Lambda'$. Recalling that $T_\lambda Z \simeq H^0(N_{\Lambda/X})$ and considering the exact sequence of the normal bundle of Λ in X we get the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_z\Lambda' \arrow[r] \arrow[d, "Id"] & T_z Z \arrow[r] \arrow[d, "d_z p"] & H^0(N_{\Lambda/X}) \arrow[r] \arrow[d, "ev"] & 0 \\ 0 \arrow[r] & T_{p(z)}\Lambda \arrow[r] & T_{p(z)}X \arrow[r] & (N_{\Lambda/X})_{p(z)} \arrow[r] & 0 \end{tikzcd}$$

which shows that $d_z p$ is surjective - ev is surjective by the spannedness of $N_{\Lambda/X}$ - hence p is smooth at z. ∎

Question 4.2. Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$ such that through every point of X there is a linear subspace of dimension s. It is possible to prove, as in [29], that the general linear subspace has a normal bundle which is spanned at the general point. Is it true that there exists a linear subspace Λ with nef normal bundle?

We end this section recalling a general construction (see [1, Proof of 0.7]):

Lemma 4.3. *Let $\Lambda \subset X \subset \mathbb{P}^N$ be a linear space contained in a smooth projective manifold such that $N_{\Lambda/X} \simeq N' \oplus \mathcal{O}(1)$ for some vector bundle N' over Λ. Then there exists a smooth hyperplane section X' of X which contains Λ and such that $N_{\Lambda/X'} \simeq N'$.*

Proof. The existence of the smooth hyperplane section follows from [8, Corollary 1.7.5]; then by the exact sequence

$$0 \longrightarrow N_{\Lambda/X'} \longrightarrow N_{\Lambda/X} \simeq N' \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}_\Lambda(1) \longrightarrow 0,$$

we obtain the statement on the normal bundle. $\square$

5. Projective n-folds covered by linear subspaces of dimension $\geq \frac{n}{2}$

In this section we will prove that, if a smooth complex projective variety $X \subset \mathbb{P}^N$ of Picard number one and dimension $2s$ contains a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$, then X is the Grassmannian of lines in $\mathbb{P}^{s+1}$.
This result, as explained in Corollary (5.2), completes [29, Main Theorem], in which smooth projective varieties of dimension n covered by linear subspaces of dimension $\geq \frac{n}{2}$ were classified.

Proposition 5.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number one and dimension $2s$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$. Then X is the Grassmannian of lines $\mathbb{G}(1, s+1)$.*

Proof. First of all notice that, by Lemma (3.2), X is a Fano manifold of index $r_X = s+1+c = s+2$ covered by lines.
Let $\sigma \colon \widetilde{X} \to X$ be the blow-up of X along Λ, denote by $E = \mathbb{P}(N^*_{\Lambda/X})$ the exceptional divisor and by $\mathcal{H}$ the pull-back $\sigma^*\mathcal{O}_X(1)$. By Proposition (3.1), $\widetilde{X}$ is a Fano manifold with a contraction $\varphi \colon \widetilde{X} \to Y$ whose restriction to E is the map associated on E to the linear system $|\xi_{N^*_{\Lambda/X}(1)}| = |\xi_{\Omega(2)}|$. This map is the $\mathbb{P}^1$-bundle over $\mathbb{G}(1,s)$ given by the projectivization of the universal quotient bundle $\mathcal{Q}$ over $\mathbb{G}(1,s)$, as shown in Example (2.3.1).
Moreover, by Proposition (3.1), $\mathrm{NE}(\widetilde{X}) = \langle [C_\sigma], [\ell] \rangle$, where C_σ is a rational curve of minimal degree contracted by σ and ℓ is the strict transform of a line meeting Λ at one point and the length of the extremal ray generated by $[\ell]$ is $r_X - n + s + 1 = 3$.
By Proposition (2.2.4) every non-trivial fiber of the contraction φ has dimension at least two. Since $E \cdot \ell = 1$ we have that E meets every non-trivial fiber of φ. As $\varphi|_E$ is equidimensional with one-dimensional fibers, it follows that φ cannot have fibers of dimension greater than two, otherwise their intersection with E would be a fiber of dimension at least two of $\varphi|_E$.
Therefore every non-trivial fiber of φ has dimension two and so, by Proposition (2.2.4), φ is of fiber type.
Let F be a general fiber of φ; by adjunction we have

$$-K_F = (-K_{\widetilde{X}})|_F = ((s+2)\mathcal{H} - (s-1)E)|_F = 3\mathcal{H}|_F,$$

hence $(F, \mathcal{H}|_F) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$.
The line bundle $2\mathcal{H} - E$ is ample and $(2\mathcal{H} - E)|_F \simeq \mathcal{O}_{\mathbb{P}^2}(1)$; thus we can apply [18, Lemma 2.12] to obtain that φ is a projective bundle over $\mathbb{G}(1, s)$.
Let $\mathcal{E} := \varphi_*\mathcal{H}$; the inclusion $E = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q}) \hookrightarrow \widetilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ gives an exact sequence of vector bundles over $\mathbb{G}(1, s)$

$$0 \longrightarrow L \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

We can compute, using the canonical bundle formula for projective bundles, that $\det \mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s)}(2)$, so, recalling that $\det \mathcal{Q} = \mathcal{O}_{\mathbb{G}(1,s)}(1)$, we have $L = \mathcal{O}_{\mathbb{G}(1,s)}(1)$.
Since $h^1(\mathcal{Q}^*(1)) = h^1(\mathcal{Q}) = 0$, the sequence splits and $\widetilde{X} = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{Q} \oplus \mathcal{O}_{\mathbb{G}(1,s)}(1))$.
We have thus proved that the existence in X of a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$ completely determines X. As the Grassmannian of lines $\mathbb{G}(1, s + 1)$ contains such a linear space - take a linear space corresponding to the lines passing through a fixed point - the proposition is proved. □

Corollary 5.2. *(cf.* [29, Main theorem]*) Let $X \subset \mathbb{P}^N$ be a smooth complex variety of dimension $n \geq 2$ covered by linear subspaces of dimensions $s \geq \frac{n}{2}$. Then X is one of the following:*

(1) *a $\mathbb{P}^r$-bundle over a smooth manifold. ($r \geq s$);*
(2) *an even dimensional hyperquadric;*
(3) *the Grassmannian of lines $\mathbb{G}(1, s + 1)$.*

Proof. In [29] the author first showed that the normal bundle of a general linear subspace is one of the following:

(i) $\mathcal{O}_{\mathbb{P}^s}^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^s}(1)^{\oplus(s-a)}$;
(ii) $\Omega_{\mathbb{P}^s}(2)$;
(iii) $T_{\mathbb{P}^s}(-1)$.

Then he showed that in case (i) X is a $\mathbb{P}^r$-bundle over a smooth manifold ($r \geq s$) and in case (ii) X is an even dimensional hyperquadric.
As for case (iii) he showed that X is the Grassmannian of lines in $\mathbb{P}^{s+1}$, if s is even or if one assumes that all the linear subspaces of the covering family have normal bundle $T_{\mathbb{P}^s}(-1)$.
Thus to prove the statement it is enough to show that, in case (iii), X is the Grassmannian of lines in $\mathbb{P}^{s+1}$; this will follow from Proposition (5.1) once we prove that, if X is as in case (iii), then its Picard number is one.
Assume that this is not the case. Since X is covered by linear subspaces, there exists a dominating family of lines in X. By adjunction we have $-K_X \cdot l \geq s + 2$, hence, by [7, Theorem 2.4], the numerical class of $[l]$ spans an extremal ray of X.
Let $\varphi \colon X \to Y$ be the contraction of this extremal ray and let F be a general fiber of φ; F has dimension at most $2s - 1$ and, by adjunction, is a Fano manifold of index $\geq s + 2$, hence $\rho_F = 1$ by [31, Theorem B].
Since F is covered by linear spaces of dimension s, applying [34, Corollary I.2.20] as in [16, Theorem 2] we derive that F is a projective space, hence the normal bundle of a general s-plane cannot be $T_{\mathbb{P}^s}(-1)$. □

Corollary 5.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1)$. Then X is the Grassmannian of lines $\mathbb{G}(1, s + 1)$.*

Proof. As in Proposition (4.1) we can prove that through every point of X there is a linear subspace of dimension s. By Corollary (5.2) X is a $\mathbb{P}^r$-bundle over a smooth manifold, an even dimensional hyperquadric or the Grassmannian of lines $\mathbb{G}(1, s+1)$. The first two cases are ruled out since the corresponding manifolds do not contain a linear subspace with normal bundle $T_{\mathbb{P}^s}(-1)$. □

6. Manifolds with Picard number one

In this section we will consider projective manifolds of dimension $2s+1$ and Picard number one containing a linear subspace of dimension s with numerically effective normal bundle, proving the following

Theorem 6.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$ and Picard number one, containing a linear subspace Λ of dimension s, such that its normal bundle $N_{\Lambda/X}$ is nef. Then X is one of the following:*

- *a linear space $\mathbb{P}^{2s+1}$;*
- *a smooth hyperquadric $\mathbb{Q}^{2s+1}$;*
- *a cubic threefold in $\mathbb{P}^4$;*
- *a complete intersection of two hyperquadrics of dimension 4 in $\mathbb{P}^5$;*
- *the intersection of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with three general hyperplanes;*
- *a hyperplane section of the Grassmannian of lines $\mathbb{G}(1, s+2)$ in its Plücker embedding.*

Proof. First of all notice that, by Lemma (3.2), X is a Fano manifold of index $r_X = s+1+c$. Moreover, all possible nef normal bundles $N_{\Lambda/X}$ are listed in Proposition (4.1).
When $N_{\Lambda/X} \simeq \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda$, X is a del Pezzo manifold with very ample fundamental divisor; hence, by the classification in [18, Theorem 8.11], of degree greater than or equal to three.
Recalling that the Picard number of X is one and that X contains lines, by the same classification we have that the degree of X is at most five.
A del Pezzo manifold of degree three and Picard number one is a cubic hypersurface in $\mathbb{P}^{n+1}$; on the other hand, by the exact sequence of normal bundles

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow \mathcal{O}_\Lambda(1)^{\oplus(s+2)} \longrightarrow \mathcal{O}_\Lambda(3) \longrightarrow 0,$$

we see that we cannot have $N_{\Lambda/X} \simeq \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda$, unless $s = 1$.
Again by [18, Theorem 8.11], a del Pezzo manifold of degree four is the complete intersection of two quadric hypersurfaces.
Let us show that also this case is possible only if $s = 1$. We owe this remark and its proof to Andrea Luigi Tironi.
Let $\mathcal{Q}$ and $\mathcal{Q}'$ be the hyperquadrics such that $X = \mathcal{Q} \cap \mathcal{Q}'$, and let $\mathcal{F}$ be the pencil generated by $\mathcal{Q}$ and $\mathcal{Q}'$; by [28, Proposition 2.1] the general quadric in $\mathcal{F}$ is smooth, so we can assume that $\mathcal{Q}$ is smooth.
By [8, Corollary 1.7.5] there is a smooth hyperplane section $\mathcal{Q}_H$ of $\mathcal{Q}$ containing Λ; by the exact sequence of normal bundles

$$0 \longrightarrow N_{\Lambda/\mathcal{Q}_H} \longrightarrow N_{\Lambda/\mathcal{Q}} \longrightarrow \mathcal{O}_\Lambda(1) \longrightarrow 0,$$

recalling that $N_{\Lambda/\mathcal{Q}_H} \simeq \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)$ we have $N_{\Lambda/\mathcal{Q}} \simeq \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)^{\oplus 2}$.
Therefore the exact sequence

$$0 \longrightarrow N_{\Lambda/X} \longrightarrow N_{\Lambda/\mathcal{Q}} \longrightarrow (N_{X/\mathcal{Q}})|_\Lambda \longrightarrow 0$$

becomes

$$0 \longrightarrow \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda \longrightarrow \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)^{\oplus 2} \longrightarrow \mathcal{O}_\Lambda(2) \longrightarrow 0.$$

A computation of the total Chern class shows that this is possible only if $s = 1$. Again by [18, Theorem 8.11], a del Pezzo manifold of degree five is a linear section of $\mathbb{G}(1, 4)$.
If $N_{\Lambda/X} \simeq \Omega_\Lambda(2) \oplus \mathcal{O}_\Lambda(1)$, then X is a smooth hyperquadric by the Kobayashi–Ochiai Theorem [21].
In case $N_{\Lambda/X} \simeq T_\Lambda(-1) \oplus \mathcal{O}_\Lambda(1)$, by Lemma (4.3) there exists a smooth hyperplane section X' of X containing Λ; moreover the normal bundle of Λ in X' is $T_\Lambda(-1)$, hence, by Corollary (5.3), X' is the Grassmannian of lines $\mathbb{G}(1, s+1)$. But, by [17, Corollary 1.3 and Proposition 2.1], $\mathbb{G}(1, s+1)$ cannot be a hyperplane section of another manifold, unless $s = 2$; in this case X' is a four-dimensional hyperquadric, hence X is a five-dimensional hyperquadric. Note that, since $T_{\mathbb{P}^2}(-1) \simeq \Omega_{\mathbb{P}^2}(2)$, this was already part of the previous case.
As to the remaining possibilities, the case $N_{\Lambda/X} \simeq T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$ is settled in subsection (6.1), while the case of a decomposable normal bundle is settled in subsection (6.2). □

6.1. **Normal bundle isomorphic to $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$.** We will start by proving that Λ belongs to a special one-dimensional family of linear subspaces of X:

Proposition 6.1.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number one and dimension $2s+1$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1) \oplus \mathcal{O}_{\mathbb{P}^s}$.*
Then there is a subvariety $\Sigma \subset X$ such that $(\Sigma, (\mathcal{O}_X(1))|_\Sigma) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1,1))$ which contains Λ as a fiber of the first projection. Moreover $\Sigma = \langle \Sigma \rangle \cap X$.

Proof. By Lemma (3.2), X is a Fano manifold of index $s+2$ covered by lines.
Let $\sigma \colon \widetilde{X} \to X$ be the blow-up of X along Λ, and denote by $E = \mathbb{P}(N^*_{\Lambda/X})$ the exceptional divisor. By Proposition (3.1), $\widetilde{X}$ is a Fano manifold with a contraction $\varphi \colon \widetilde{X} \to Y$ whose restriction to E is the map associated on E to the linear system $|\xi_{N^*_{\Lambda/X}(1)}| = |\xi_{\Omega(2)\oplus\mathcal{O}(1)}|$, i.e., the blow-up of $\mathbb{G}(1, s+1)$ along a sub-Grassmannian $\mathbb{G}(1, s)$ as shown in Example (2.3.3). By Proposition (3.1) we also have that the extremal ray associated with φ is generated by the class $[\ell]$ of the strict transform of a line $l \subset X$ meeting Λ at one point and has length 2. Let $\mathcal{H}$ be the pull-back $\sigma^*\mathcal{O}_X(1)$. We can take $K_{\widetilde{X}} + 2\mathcal{H}$ as a supporting divisor for φ.
Since $E \cdot \ell = 1$, we have that E meets every non-trivial fiber of φ.
As $\varphi|_E$ is equidimensional with one-dimensional fibers, it follows that φ cannot have fibers of dimension greater than two, otherwise their intersection with E would be a fiber of dimension at least two of $\varphi|_E$. Therefore every non-trivial fiber of φ has dimension at most two.

We claim that φ is of fiber type. Assume by contradiction that φ is birational. Then it is equidimensional by Proposition (2.2.4). We can apply [3, Theorem 4.1] to get that Y is smooth and φ is the blow-up of a smooth codimension-three center T. Since E meets every non-trivial fiber of φ we have $T = \mathbb{G}(1, s)$.
So Y contains $\varphi(E) = \mathbb{G}(1, s+1)$ as an effective divisor, but, since $\rho_Y = 1$, this implies that $\mathbb{G}(1, s+1)$ is ample in Y; it thus follows by [17, Corollary 1.3 and Proposition 2.1] that $s = 2$.

Therefore Y is a projective space or a smooth hyperquadric and $T \simeq \mathbb{P}^2$. Using the two different blow-up structures of $\widetilde{X}$ we can write

$$4\mathcal{H} - 2E = -K_{\widetilde{X}} = -\varphi^* K_Y - 2\,\mathrm{Exc}(\varphi).$$

Therefore the index of Y is even, so $Y \simeq \mathbb{P}^5$; but the blow-up of $\mathbb{P}^5$ along $\mathbb{P}^2$ has one fiber type contraction, so also this case cannot happen.

It follows that φ is of fiber type. Since E meets every fiber of φ we have $Y = \mathbb{G}(1, s+1)$. The restriction of φ to E is birational, hence the general fiber of φ has dimension one. As already noticed, any fiber of φ has dimension at most two, hence φ is a special Bănică scroll, so $\widetilde{X} = \mathbb{P}_{\mathbb{G}(1,s+1)}(\mathcal{E})$, where $\mathcal{E} := \varphi_*\mathcal{H}$.
Combining the canonical bundle formula for $\widetilde{X}$ as a blow-up with the canonical bundle formula for $\widetilde{X}$ as a Bănică scroll we get

$$-s(\mathcal{H}-E) = K_{\widetilde{X}} + 2\mathcal{H} = \varphi^*(K_{\mathbb{G}(1,s+1)} + \det\mathcal{E}) = \varphi^*\mathcal{O}_{\mathbb{G}(1,s+1)}(-s-2+\deg\det\mathcal{E}),$$

hence $\det\mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s+1)}(2)$.
Denote by Λ_0 the section of $\sigma\colon E \to \Lambda$ which corresponds to the surjection $\Omega_{\mathbb{P}^s}(1) \oplus \mathcal{O}_{\mathbb{P}^s} \to \mathcal{O}_{\mathbb{P}^s}$; the restriction of $\xi_{N^*_{\Lambda/X}(1)}$ to Λ_0 is $\mathcal{O}_{\Lambda_0}(1)$, hence Λ_0 is mapped to a linear subspace Λ_1 of $\mathbb{G}(1, s+1)$.
Let l be any line in Λ_1; the line in Λ_0 mapped to l is a section corresponding to a surjection $\mathcal{E}|_l \to \mathcal{O}_l(1)$; hence $\mathcal{E}|_l \simeq \mathcal{O}_l(1)^{\oplus 2}$. By [14, Théorème] the restriction of $\mathcal{E}$ to Λ_1 is decomposable: $\mathcal{E}|_{\Lambda_1} \simeq \mathcal{O}_{\Lambda_1}(1)^{\oplus 2}$.
So $(\mathbb{P}_{\Lambda_1}(\mathcal{E}|_{\Lambda_1}), \mathcal{H}) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^s}(1,1))$, and $\Sigma := \sigma(\mathbb{P}_{\Lambda_1}(\mathcal{E}|_{\Lambda_1}))$ is a subvariety such that $(\Sigma, \mathcal{O}_X(1)) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^s}(1,1))$ which contains Λ as a fiber of the first projection.
To prove the last assertion note that Σ is the base locus of the linear subsystem of $|\mathcal{O}_X(1) \otimes \mathcal{I}_\Lambda|$ given by the pull-back of the linear system $|\mathcal{O}_{\mathbb{G}(1,s+1)}(1) \otimes \mathcal{I}_{\Lambda_1}|$. □

Now we will determine the normal bundle in X of the subvariety Σ constructed in the previous proposition.

Proposition 6.1.2. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number one and dimension $2s+1$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_\Lambda(-1) \oplus \mathcal{O}_\Lambda$. Let $\Sigma \subset X$ be as in Proposition (6.1.1).*
Then $N_{\Sigma/X} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* T_{\mathbb{P}^s}(-1)$, where p_1 and p_2 denote the projections of $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^s$ onto the factors.

Proof. By Lemma (3.2), X is a Fano manifold of index $s+2$ covered by lines. Let $\sigma\colon \widetilde{X} \to X$ be the blow-up of X along Σ, and denote by $E = \mathbb{P}(N^*_{\Sigma/X})$ the exceptional divisor. By Proposition (3.1), E is a Fano manifold. By adjunction

$$\det N_{\Sigma/X} = K_\Sigma - (K_X)|_\Sigma = \mathcal{O}_\Sigma(s, 1)$$

Let $p\colon E \to \mathbb{P}^s$ be the composition of the bundle projection with p_2; the fiber F_x of p over a point $x \in \mathbb{P}^s$ is $\mathbb{P}_{l_x}((N^*_{\Sigma/X})|_{l_x})$ where l_x is the fiber of p_2 over x.
By adjunction F_x is a Fano manifold, hence, recalling that $c_1((N^*_{\Sigma/X})|_{l_x}) = -s$, we have that $(N^*_{\Sigma/X})|_{l_x} \simeq \mathcal{O}_{l_x}(-1)^{\oplus s}$.
So $N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1)$ is trivial on the fibers of p_2, hence $N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) = p_2^*(\mathcal{F})$, with $\mathcal{F}$ a vector bundle on $\mathbb{P}^s$. In particular

$$(N_{\Sigma/X})|_\Lambda \simeq (N_{\Sigma/X} \otimes p_1^*\mathcal{O}_{\mathbb{P}^1}(-1))|_\Lambda \simeq (p_2^*(\mathcal{F}))|_\Lambda = \mathcal{F}.$$

From the the exact sequence of normal bundles

$$0 \longrightarrow \mathcal{O}_\Lambda \longrightarrow T_\Lambda(-1) \oplus \mathcal{O}_\Lambda \longrightarrow (N_{\Sigma/X})|_\Lambda \longrightarrow 0,$$

it follows that $(N_{\Sigma/X})|_\Lambda$ is nef; recalling that $c_1((N_{\Sigma/X})|_\Lambda) = 1$ we have that $(N_{\Sigma/X})|_\Lambda$ is uniform so, either $(N_{\Sigma/X})|_\Lambda$ is decomposable or $(N_{\Sigma/X})|_\Lambda \simeq T_{\mathbb{P}^s}(-1)$. The first case is not possible, since the sequence would split. Therefore $\mathcal{F} = T_{\mathbb{P}^s}(-1)$ and $N_{\Sigma/X} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* T_{\mathbb{P}^s}(-1)$. □

Now we prove that the existence of a subvariety Σ as above completely determines the manifold X.

Proposition 6.1.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number one and dimension $2s+1$ containing a subvariety Σ such that $\Sigma = \langle \Sigma \rangle \cap X$ and $(\Sigma, (\mathcal{O}_X(1))|_\Sigma)) \simeq (\mathbb{P}^1 \times \mathbb{P}^s, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^s}(1,1))$, with $N_{\Sigma/X} \simeq p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^* T_{\mathbb{P}^s}(-1)$. Let $\sigma\colon \widetilde{X} \to X$ be the blow-up of X along Σ. Then $\widetilde{X}$ is a divisor in the linear system $|2\xi - \varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(1)|$ in $\mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{O}_{\mathbb{G}(1,s)}(1) \oplus \mathcal{Q}^{\oplus 2})$, where ξ denotes the tautological line bundle and φ the bundle projection.*

Proof. By Lemma (3.2) X is a Fano manifold of index $s+2$ covered by lines. Denote by $E = \mathbb{P}(N^*_{\Sigma/X})$ the exceptional divisor; by Proposition (3.1), $\widetilde{X}$ and E are Fano manifolds. Moreover, the ray associated with the extremal contraction $\varphi\colon \widetilde{X} \to Y$ different from σ has length three and its restriction to E is the map associated on E to the linear system $|\xi_{N^*_{\Sigma/X}(1)}| = |\xi_{p_2^*\Omega_{\mathbb{P}^s}(2)}|$ described in Example (2.3.2). Denote by $\mathcal{H}$ the pull-back $\sigma^*\mathcal{O}_X(1)$; we can take $K_{\widetilde{X}} + 3\mathcal{H}$ as a supporting divisor for φ. Since $E \cdot \ell = 1$ we have that E meets every non-trivial fiber of φ. As $\varphi|_E$ is equidimensional with two-dimensional fibers, it follows that φ cannot have fibers of dimension greater than three, otherwise their intersection with E would be a fiber of dimension at least three of $\varphi|_E$. Therefore every non-trivial fiber of φ has dimension at most three.

We claim that φ is of fiber type. Assume by contradiction that φ is birational; then by Proposition (2.2.4) it is equidimensional. We can apply [3, Theorem 4.1] to get that Y is smooth and φ is the blow-up of Y along a smooth center.
Since $E \cdot \ell = 1$ the intersection of E with a fiber of φ is a $\mathbb{P}^2$, contradicting the fact that fibers of $\varphi|_E$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

It follows that φ is of fiber type. Since E meets every fiber of φ we have $Y = \mathbb{G}(1,s)$. The contraction φ is supported by $K_{\widetilde{X}} + 3\mathcal{H}$, it is elementary and equidimensional with three-dimensional fibers, so it is a quadric bundle.
Let $\mathcal{E} := \varphi_*\mathcal{H}$; $\widetilde{X}$ embeds in $P = \mathbb{P}_{\mathbb{G}(1,s)}(\mathcal{E})$ as a divisor of relative degree 2.
Let $\mathcal{E}' := \varphi_*(\mathcal{H}|_E)$; notice that, as shown in Example (2.3.2), $\mathcal{E}' \simeq \mathcal{Q}^{\oplus 2}$.
The vector bundle $\mathcal{E}$ has $\mathcal{E}'$ as a quotient. Indeed, if $x \in \mathbb{G}(1,s)$ is a point and we denote by F and f the fibers of φ and $\varphi|_E$ over x, we have that $\mathcal{E}'_x = H^0(\mathcal{H}|_f)$ is a quotient of $\mathcal{E}_x = H^0(\mathcal{H}|_F)$.
It follows that there exists an exact sequence on $\mathbb{G}(1,s)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{G}(1,s)}(a) \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \oplus \mathcal{Q} \longrightarrow 0.$$

Call P' the projectivization of $\mathcal{E}'$; since $\widetilde{X}|_{P'} = E$ we have, by Example (2.3.2), that

$$\widetilde{X} = 2\mathcal{H} - \varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(1).$$

Combining the canonical bundle formula for P,

$$K_P + 5\mathcal{H} = \varphi^*(K_{\mathbb{G}(1,s)} + \det \mathcal{E}),$$

with the blow-up formula giving the canonical bundle of $\widetilde{X}$

$$K_{\widetilde{X}} = -(s+2)\mathcal{H} + (s-1)E$$

and the adjunction formula

$$K_{\widetilde{X}} = (K_P + \widetilde{X})|_{\widetilde{X}},$$

we obtain

$$\begin{aligned} -(s+2)\mathcal{H} + (s-1)E &= -5\mathcal{H} + \varphi^*(K_{\mathbb{G}(1,s)} + \det \mathcal{E}) + 2\mathcal{H} - \varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(1) \\ &= -3\mathcal{H} + \varphi^*\mathcal{O}_{\mathbb{G}(1,s)}(-s-2+\deg\det\mathcal{E}) \\ &= -3\mathcal{H} + (-s-2+\deg\det\mathcal{E})(\mathcal{H}-E) \\ &= (-s-5+\deg\det\mathcal{E})\mathcal{H} + (s+2+\deg\det\mathcal{E})E. \end{aligned}$$

It follows that $\deg\det\mathcal{E} = 3$, therefore $a = 1$; since $h^1(\mathcal{Q}^*(1)^{\oplus 2}) = h^1(\mathcal{Q}^{\oplus 2}) = 0$, the above sequence splits and $\mathcal{E} = \mathcal{O}_{\mathbb{G}(1,s)}(1) \oplus \mathcal{Q}^{\oplus 2}$. □

Corollary 6.1.4. *Let $X \subset \mathbb{P}^N$ be a smooth variety of Picard number one and dimension $2s+1$ containing a linear subspace $\Lambda \simeq \mathbb{P}^s$ whose normal bundle is $T_{\mathbb{P}^s}(-1) \oplus \mathcal{O}_{\mathbb{P}^s}$; then X is a hyperplane section of the Grassmannian of lines $\mathbb{G}(1, s+2)$.*

Proof. By Propositions (6.1.1), (6.1.2), (6.1.3), there is only one manifold which contains a linear subspace as in the statement. A smooth hyperplane section $\mathbb{G}(1, s+2) \cap H$ of the Grassmannian of lines $\mathbb{G}(1, s+2)$ contains such a linear space - just take the intersection of H with a linear space corresponding to lines passing through a fixed point - so the statement follows. □

6.2. **Decomposable normal bundle.**

Proposition 6.2.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $n \geq 4$ containing a linear space Λ of dimension s with $\left[\frac{n}{2}\right] \leq s \leq n-2$. Assume that the normal bundle $N_{\Lambda/X}$ is trivial. Then the Picard number of X is at least two.*

Proof. Assume by contradiction that $\rho_X = 1$; by Lemma (3.2), X is a Fano manifold of index $r_X = s+1$ covered by lines. By the first part of Proposition (3.1) the blow-up of X along Λ, which we will denote by $\widetilde{X}$, is a Fano manifold with $\rho_{\widetilde{X}} = 2$, whose "other" contraction φ is given by the linear system $|\mathcal{H} - E|$, where $\mathcal{H} := \sigma^*\mathcal{O}_X(1)$. The restriction of $\mathcal{H} - E$ to $E = \Lambda \times \mathbb{P}^{n-s-1}$ is $\xi_{N^*_{\Sigma/X}(1)} = \mathcal{O}_E(1,1)$. In particular no curves of E are contracted by φ.
The extremal ray associated with φ is generated by the class $[\ell]$ of the strict transform of a line meeting Λ at one point, hence $E \cdot \ell = 1$. Since E has positive intersection number with curves contracted by φ, it follows that every non-trivial fiber of φ has dimension one. By Proposition (3.1) the extremal ray associated with φ has length $r_X - n + s + 1 = 2s - n + 2$, hence, by Proposition (2.2.4), every non-trivial fiber has dimension at least $2s - n + 1$. It follows that $s = \left[\frac{n}{2}\right]$.

If n is even, then the extremal ray associated with φ has length 2, hence φ is of fiber type and its general fiber is $\mathbb{P}^1$. The line bundle E restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on a general fiber of φ, so we can apply [18, Lemma 2.12] and obtain that φ makes X

into a $\mathbb{P}^1$-bundle over Y.
The divisor E is a regular section of this $\mathbb{P}^1$-bundle, hence $E \simeq Y$, contradicting the fact that $\rho_Y = 1$.

If n is odd, then the extremal ray associated with φ has length 1.

If φ is of fiber type, then it is a conic bundle; by [2, Theorem B] Y is smooth. The smooth variety Y has Picard number one and it is dominated by the toric variety $E = \mathbb{P}^s \times \mathbb{P}^s$, hence, by [26, Theorem 1], Y is a projective space $\mathbb{P}^{2s}$. Since $n \geq 4$ we have $2s \geq 3$, hence ρ_E must be one by [23, Corollary 3.2], a contradiction.

If φ is birational, then Y is smooth and φ is the blow-up of a smooth codimension two subvariety. The divisor $D = (s+2)(\mathcal{H} - E) = \varphi^*\mathcal{O}_Y(s+2)$ is nef and big on $\widetilde{X}$. Therefore $h^1(\mathcal{H} - 2E) = h^1(K_{\widetilde{X}} + D) = 0$; hence the morphism

$$H^0(\widetilde{X}, \mathcal{H} - E) \longrightarrow H^0(E, (\mathcal{H} - E)|_E) = H^0(E, \xi_{N^*_{\Sigma/X}(1)})$$

is surjective, so the restriction to E of φ is the morphism given by the linear system $|\xi_{N^*_{\Sigma/X}(1)}| = |\mathcal{O}_E(1,1)|$. In particular the image of E via φ is a smooth divisor isomorphic to $\mathbb{P}^s \times \mathbb{P}^s$. Since $\rho_Y = 1$ this is impossible, by [30, Proposition IV]. □

Remark 6.2.2. If $n = 3$, then, by Lemma (3.2), X is a Fano manifold of index 2 covered by lines, hence a del Pezzo manifold. By the classification in [18, Theorem 8.11], a del Pezzo threefold of Picard number one with very ample fundamental divisor is a cubic hypersurface in $\mathbb{P}^4$, the complete intersection of two hyperquadrics in $\mathbb{P}^5$ o a linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with three general hyperplanes.

Proposition (6.2.1) allows us to prove that, if the normal bundle of Λ is decomposable, the Picard number of X is one and $s \geq 2$, then X is a linear space.

Corollary 6.2.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and Picard number one containing a linear space Λ of dimension $s \geq 2$ with $\left[\frac{n}{2}\right] \leq s \leq n-2$. Assume that the normal bundle $N_{\Lambda/X}$ is $\mathcal{O}_\Lambda(1)^{\oplus c} \oplus \mathcal{O}_\Lambda^{\oplus(n-s-c)}$. Then $c = n - s$ and X is a linear space.*

Proof. By Proposition (6.2.1) we can assume that $c > 0$, so, by Lemma (4.3) we can find a smooth hyperplane section X' of X containing Λ. Then, as in [16, Theorem 2], we apply [34, Corollary I.2.20], which yields that X' is a linear space, so we conclude that X is a linear space, too. □

7. Manifolds with Picard number greater than one

Let X be a smooth complex projective variety, let V be an unsplit dominating family of rational curves for X and let $q\colon X \dashrightarrow Y$ be the rc(V)-fibration. Let B be the indeterminacy locus of q; notice that $\dim B \leq \dim X - 2$, as X is smooth. Moreover, by [10, Proposition 1] B is the union of all rc(V)-equivalence classes of dimension greater than $\dim X - \dim Y$.

Lemma 7.1. *Let V be an unsplit dominating family of rational curves on a smooth projective variety X. Let B be the indeterminacy locus of the rc(V)-fibration $q\colon X \dashrightarrow Y$, let D be a very ample divisor on $q(X \setminus B)$ and let $\widehat{D} := \overline{q^{-1}D}$. Then*

(a) $\widehat{D} \cdot V = 0$;

(b) *if* $C \not\subset B$ *is a curve whose numerical class is not proportional to* $[V]$*, then* $\widehat{D} \cdot C > 0$*;*
(c) *if* $[V]$ *does not span an extremal ray of* $\mathrm{NE}(X)$*, then there exists a curve* $C \subset B$ *whose class is not proportional to* $[V]$ *such that* $\widehat{D} \cdot C \leq 0$*.*

Proof. A general cycle of V is contained in a fiber of q disjoint from $\widehat{D}$, so $\widehat{D} \cdot V = 0$. If C is as in (b), then $q(C)$ is a curve in Y and the result follows from projection formula.
Finally, if $[V]$ does not span an extremal ray, then either $\widehat{D}$ is not nef or $\widehat{D}$ is nef but $\widehat{D}^{\perp} \cap \mathrm{NE}(X) \supsetneq [V]$. In both cases there exists a curve $C \subset X$ whose class is not proportional to $[V]$ such that $\widehat{D} \cdot C \leq 0$. Such a curve must be contained in B by [10, Proof of Proposition 1]. $\square$

Lemma 7.2. *Let* X *be a manifold which admits an unsplit dominating family of rational curves* V*. Assume that there exists an extremal face* $\Sigma \subseteq \overline{\mathrm{NE}(X)}_{K_X<0}$ *such that* $[V] \subset \Sigma$*.*
Then, either $[V]$ *spans an extremal ray, or there exists an extremal ray in* Σ *whose exceptional locus is contained in the indeterminacy locus* B *of the* $\mathrm{rc}(V)$*-fibration. In particular, this ray is associated with a small contraction.*

Proof. Let τ be a minimal subface of Σ containing $[V]$. If $\dim \tau = 1$, then $[V]$ spans an extremal ray.
Assume that $\dim \tau \geq 2$. Let $\widehat{D}$ be as in Lemma (7.1). Since $\widehat{D} \cdot V = 0$, then either $\widehat{D}$ is zero on every extremal ray of σ or it is negative on at least one ray. In both cases, by part (b) of Lemma (7.1) there is at least one ray whose exceptional locus is contained in B, and the assertion follows as $\dim B \leq \dim X - 2$. $\square$

The following is a slight improvement of [9, Theorem 2.5] (cf. also [7, Theorem 2.4], where the case $-K_X \cdot V \geq \frac{n+3}{2}$ is treated):

Theorem 7.3. *Let* (X, H) *be a polarized* n*-fold with a dominating family of rational curves* V *such that* $H \cdot V = 1$*. If* $-K_X \cdot V \geq \frac{n+1}{2}$*, then* $[V]$ *spans an extremal ray of* $\mathrm{NE}(X)$*.*

Proof. Denote by m the positive integer $-K_X \cdot V$ and by L the adjoint divisor $K_X + mH$.

Assume first that L is nef.
Denote by $q \colon X \dashrightarrow Y$ the $\mathrm{rc}(V)$-fibration and by B its indeterminacy locus. Assume that $[V]$ does not span an extremal ray in $\mathrm{NE}(X)$. This implies that L defines an extremal face Σ of dimension at least two, containing $[V]$.
By Lemma (7.2) there exists an extremal ray $R \in \Sigma$ whose associated contraction φ is small; moreover, since $L \cdot R = 0$ the length of this extremal ray is greater than or equal to m.
If F is a non-trivial fiber of φ, by Proposition (2.2.4), we have $\dim F \geq m + 1$.
Let x be a point in F; $\mathrm{Locus}(V_x)$ meets F, but, since $[V]$ is independent from R, the intersection has to be zero-dimensional. This implies that

$$\dim \mathrm{Locus}(V_x) \leq n - m - 1 \leq \frac{n-3}{2},$$

contradicting part (b) of Proposition (2.2.3).

Assume now that L is not nef.
This assumption yields the existence of an extremal ray R such that $L \cdot R < 0$. Notice that R has length $\geq m+1$, hence every non-trivial fiber of the associated contraction has dimension $\geq m$ by Proposition (2.2.4).
We have, by Lemma (2.2.6),

$$\dim X \geq \dim \mathrm{Locus}(V)_F \geq -K_X \cdot V + \dim F - 1 \geq m + m - 1 \geq n,$$

hence $\mathrm{Locus}(V)_F = X$. We can apply the second part of Lemma (2.2.6) to get $\mathrm{NE}(X) = \langle [V], R \rangle$ and we are done. □

Remark 7.4. Combining the ideas and tecniques of [9] with those of [10] it is actually possible to prove the statement of Theorem (7.3) under the weaker assumption that $-K_X \cdot V \geq \frac{n-1}{2}$; however the proof becomes very long and complicated, so, since it is not necessary for our main theorem, we will present it elsewhere ([24]).

Theorem 7.5. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension $2s+1$ and Picard number greater than one, containing a linear subspace Λ of dimension s, whose normal bundle $N_{\Lambda/X}$ is nef. Then there is an elementary contraction $\varphi\colon X \to Y$ which contracts Λ. Moreover one of the following occurs:*

- *φ is a scroll;*
- *Y is a curve, and the general fiber of φ is*
  - *the Grassmannian of lines $\mathbb{G}(1, s)$;*
  - *a smooth hyperquadric $\mathbb{Q}^{2s}$;*
  - *a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$.*

Proof. By the proof of Proposition (4.1), there is a component T of the Hilbert scheme of linear spaces of dimension s in X containing the point corresponding to Λ such that through any point of X there is a linear space of dimension s parametrized by T.
Therefore, denoted by l a general line in Λ there is a dominating family of lines in X containing l. By adjunction we have $-K_X \cdot l \geq s+1$, hence, by Theorem (7.3), the numerical class of l spans an extremal ray of X.
Let $\varphi\colon X \to Y$ be the contraction of this extremal ray and let F be a general fiber of φ; F has dimension at most $2s$ and, by adjunction, is a Fano manifold of index $\geq s+1$, hence either $F \simeq \mathbb{P}^s \times \mathbb{P}^s$ or $\rho_F = 1$ by [31, Theorem B].
Assume that $\rho_F = 1$; since F is covered by linear spaces of dimension s, then, by Corollary (5.2), F is a projective space, a hyperquadric of dimension $2s$ or a Grassmannian of lines $\mathbb{G}(1, s)$. □

Remark 7.6. If φ is a scroll and $\dim F \geq s+1$, then X has a projective bundle structure over Y by [13, Theorem 1.7]. By [8, Conjecture 14.1.10] this should be the case also if $\dim F = s$.

Example 7.7. We show with an example that the last case of Theorem (7.5) is effective; the idea on which it is based has been suggested by Wiśniewski for [25, Example 7.2].
Let C' be a smooth curve with a free $\mathbb{Z}_2$-action, so that the action induces an étale covering $\pi\colon C' \to C$ of degree 2. Let G be $\mathbb{P}^s \times \mathbb{P}^s$ and take on G the $\mathbb{Z}_2$-action which exchanges the factors.
Let $X' = G \times C'$ and let X be the quotient of X' by the product action of $\mathbb{Z}_2$; the

action is free and so X is smooth. By the universal property of group actions there exists a morphism $\varphi\colon X \to C$ such that the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\pi'"'] & C' \arrow[d, "\pi"] \\ X \arrow[r, "\varphi"] & C \end{tikzcd}$$

The map $\varphi\colon X \to C$ is an extremal contraction and every fiber is a product of projective spaces $\mathbb{P}^s \times \mathbb{P}^s$. We will now show that φ is elementary.
Let l be a line in G and consider the product $l \times C' \subset G \times C' = X'$: it is a flat family of rational curves. Let c be a point of C and let $\{c'_1, c'_2\}$ be $\pi^{-1}(c)$; finally set $l'_i = l \times \{c'_i\}$ and consider the restriction of the previous diagram

$$\begin{tikzcd} G \times \{c'_1, c'_2\} \arrow[r] \arrow[d, "\pi'"'] & \{c'_1, c'_2\} \arrow[d, "\pi"] \\ \varphi^{-1}(c) \simeq \mathbb{P}^s \times \mathbb{P}^s \arrow[r, "\varphi"] & c \end{tikzcd}$$

Since the product action identifies $G \times \{c'_1\}$ with $G \times \{c'_2\}$ exchanging the factors we have that $l_1 = \pi(l'_1)$ is a line in a fiber of the projection of $\varphi^{-1}(c)$ onto the first factor and $l_2 = \pi(l'_2)$ is a line in a fiber of the projection of $\varphi^{-1}(c)$ onto the second factor, hence lines in the two factors are algebraically and thus numerically equivalent.

Remark 7.8. In the last example, X has an unsplit dominating family of rational curves V such that V_x has dimension $\frac{\dim X - 3}{2}$ and is reducible for every x.
This should be compared with [20, Theorem 5.1], in which it is proved that, if $\dim V_x \geq \frac{\dim X - 1}{2}$, then V_x is irreducible.

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